

NEXT IAS

(To be filled by candidate)

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Roll No. :

Registration Number : Date of Examination : 30/06/2023

Exam Centre : Old Rajinder Nagar ☐

Bhopal ☐

Online ☒

CSE (MAINS) TEST SERIES - 2023

Test-01

MATHEMATICS
OPTIONAL

Dated : 18-06-2023

Time Allowed : Three Hours

Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are **EIGHT** questions divided in **Two** sections and printed in **ENGLISH**.

Candidate has to attempt **FIVE** questions in all.

Question No. 1 and 5 are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from each section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA). Booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order. Unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA). Booklet must be clearly struck off.

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SECTION-A

Q.1 (a) Let $u_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix}$, $u_2 = \begin{pmatrix} 2 \\ -1 \\ -5 \\ 2 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 \\ -1 \\ -4 \\ 0 \end{pmatrix}$. Find the value of h so that $b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ h \end{pmatrix}$ lie in span $\{u_1, u_2, u_3\}$. (10)

if $b \in \text{span}\{u_1, u_2, u_3\}$ then b can be written as

$$b = xu_1 + yu_2 + zu_3 \quad \text{for some } x, y, z$$

i.e.
$$\begin{bmatrix} 2 \\ 1 \\ 1 \\ h \end{bmatrix} = \begin{bmatrix} x + 2y + z \\ x - y - z \\ 2x - 5y - 4z \\ 4x + 2y \end{bmatrix}$$

thus to find value of h we need to find solutions of following equations

$$x + 2y + z = 2$$

$$x - y - z = 1$$

$$2x - 5y - 4z = 1$$

$$4x + 2y = h$$

consider

$$A.M. = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & h \end{array} \right]$$

C.M.

Applying $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - 2R_1$
 $R_4 \rightarrow R_4 - 4R_1$

$$A.M.N \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & h-8 \end{array} \right]$$

Applying $R_3 \rightarrow R_3 / -3$
 $R_4 \rightarrow R_4 / -2$

$$A.M.N \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 3 & 2 & 1 \\ 0 & 3 & 2 & \frac{h-8}{-2} \end{array} \right]$$

Applying $R_3 \rightarrow R_3 + R_2$
 $R_4 \rightarrow R_4 + R_2$

$$A.M.N \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{h-8}{-2} - 1 \end{array} \right]$$

C.M.

Applying $R_3 \leftrightarrow R_4$

$$A.M.N \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & h-6/-2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

C.M.

for solution to exist $\text{Rank}(AM) = \text{Rank}(CM)$
 $= 2$

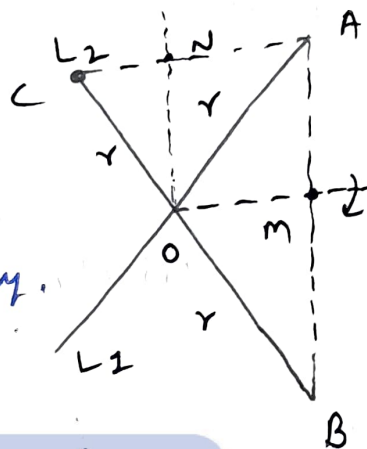
thus $\frac{h-6}{-2} = 0$

$$\boxed{h=6}$$

Q.1 (b)

(l_1, m_1, n_1) and (l_2, m_2, n_2) are the dc's of two concurrent lines, show that the dc's of the two lines bisecting the angles between them are proportional to $(l_1 \pm l_2, m_1 \pm m_2, n_1 \pm n_2)$. (10)

Let L_1 and L_2 be two concurrent lines with dc's (l_1, m_1, n_1) and (l_2, m_2, n_2) respectively.



Consider O be the origin

Now we draw OA and OB

parallel to given lines such that

$$OA = OB = r$$

then coordinates of A = $(l_2 r, m_2 r, n_2 r)$

coordinates of B = $(l_1 r, m_1 r, n_1 r)$

and coordinates of C = $(-l_2 r, -m_2 r, -n_2 r)$

Let M and N be the middle point of AB and AC respectively. then the

coordinates of M and N are

$$M \left(\frac{l_1 r + l_2 r}{2}, \frac{m_1 r + m_2 r}{2}, \frac{n_1 r + n_2 r}{2} \right)$$

$$\text{and } N \left(\frac{l_1 r - l_2 r}{2}, \frac{m_1 r - m_2 r}{2}, \frac{n_1 r - n_2 r}{2} \right)$$

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Since OM and ON are internal and external bisector of $\angle AOB$.

Hence Direction ratios of OM and ON are ~

$$(d_1 \pm d_2, m_1 \pm m_2, n_1 \pm n_2)$$

Hence proved



Q.1(c)

If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, find $A^7 - 7A^6 + 15A^5 - 12A^4 - A^3 + 8A^2 - 15A - 4I$. (10)

characteristic equation of A is given by,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda) \{ (2-\lambda)^2 - 1 \} + 1 \{ (-2+\lambda) + 1 \} + 1 \{ 1 - (2-\lambda) \} = 0$$

$$(2-\lambda) \{ 4 + \lambda^2 - 4\lambda - 1 \} + \{ -1 + \lambda \} + \{ -1 + \lambda \} = 0$$

$$(2-\lambda) (\lambda^2 - 4\lambda + 3) - 2 + 2\lambda = 0$$

$$2\lambda^2 - 8\lambda + 6 - \lambda^3 + 4\lambda^2 - 3\lambda - 2 + 2\lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

$$\cancel{(\lambda-1)} \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \quad \text{--- (1)}$$

We know by Cayley's - Hamilton theorem that every square matrix satisfies its own characteristic equation.

Hence by eqn. ①

$$A^3 - 6A^2 + 9A - 4I = 0 \quad \text{--- ②}$$

We need to find

$$\begin{aligned} A^7 - 7A^6 + 15A^5 - 12A^4 - A^3 + 8A^2 - 15A - 4I \\ = (A^4 - A^3 + A + I)(A^3 - 6A^2 + 9A - 4I) \\ + (5A^2 - 20A) \end{aligned}$$

from ②

$$\begin{aligned} A^7 - 7A^6 + 15A^5 - 12A^4 - A^3 + 8A^2 - 15A - 4I \\ = 5A^2 - 20A \quad \text{--- ③} \end{aligned}$$

NOW $A^2 = A \cdot A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

then $5A^2 - 20A$

$$= 5 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 40 & -20 & 20 \\ -20 & 40 & -20 \\ 20 & -20 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} 30 & -25 & 25 \\ -25 & 30 & -25 \\ 25 & -25 & 30 \end{bmatrix} + \begin{bmatrix} -40 & +20 & -20 \\ 20 & -40 & 20 \\ -20 & 20 & -40 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & -5 & 5 \\ -5 & -10 & -5 \\ 5 & -5 & -10 \end{bmatrix}$$

// Answer

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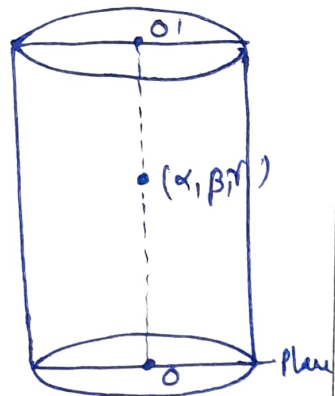
Q.1(d)

Find the equation of the right circular cylinder which passes through the circle
 $x^2 + y^2 + z^2 = 9, x - y + z = 3$.
 (10)

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Given circle

$$\begin{aligned} S: x^2 + y^2 + z^2 &= 9 \\ P: x - y + z &= 3 \end{aligned} \quad \text{--- (1)}$$



Since given circle is the
base of right circular

cylinder hence axis of cylinder

OO' is perpendicular to the plane P.

thus equation of axis OO' passing through
point (α, β, γ) is

$$\frac{x - \alpha}{1} = \frac{y - \beta}{-1} = \frac{z - \gamma}{1} = r \quad \text{--- (2)}$$

general point on this line is $(r + \alpha, -r + \beta, r + \gamma)$

this point lies on circle (1) is

$$\therefore (r + \alpha)^2 + (-r + \beta)^2 + (r + \gamma)^2 = 9 \quad \text{--- (3)}$$

$$r + \alpha + r - \beta + r + \gamma = 3 \quad \text{--- (4)}$$

$$3r = 3 - \alpha + \beta - \gamma$$

$$r = \frac{3 - \alpha + \beta - \gamma}{3}$$

Putting this value of r in equation (3) is

$$\left\{ \frac{3-\alpha+\beta-\gamma}{3} + \alpha \right\}^2 + \left\{ \frac{-3+\alpha-\beta+\gamma}{3} + \beta \right\}^2 +$$

$$\left\{ \frac{3-\alpha+\beta-\gamma}{3} + \gamma \right\}^2 = 9$$

$$\left\{ \frac{3+2\alpha+\beta-\gamma}{3} \right\}^2 + \left\{ \frac{\alpha+2\beta+\gamma-3}{3} \right\}^2 +$$

$$\left\{ \frac{-\alpha+\beta+2\gamma+3}{3} \right\}^2 = 9$$

$$9 + 4\alpha^2 + 12\alpha + \beta^2 + \gamma^2 - 2\beta\gamma + 2(3+2\alpha)(\beta+\gamma)$$

$$+ \alpha^2 + 4\beta^2 + 4\alpha\beta + \gamma^2 + 9 - 6\gamma + 2(\alpha+2\beta)(\gamma-3)$$

$$+ \alpha^2 + \beta^2 - 2\alpha\beta + 4\gamma^2 + 9 + 12\gamma = 81$$

$$+ 2(-\alpha+\beta)(2\gamma+3)$$

$$6(\alpha^2 + \beta^2 + \gamma^2) + 12\alpha - 2\beta\gamma + 6\beta - 6\gamma + 4\alpha\beta$$

$$- 4\alpha\gamma + 4\alpha\beta - 6\gamma + 2\alpha\gamma - 6\alpha + 4\beta\gamma + 4\beta\gamma$$

$$6\beta - 4\alpha\gamma - 6\alpha - 12\beta - 2\alpha\beta + 12\gamma = 54$$

$$6(\alpha^2 + \beta^2 + \gamma^2) + 6\alpha\beta + 6\beta\gamma - 6\alpha\gamma = 54$$

$$\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \beta\gamma - \alpha\gamma = 9$$

locus of (α, β, γ) is

$$\boxed{x^2 + y^2 + z^2 + xy + yz - zx = 9}$$

Q.1 (c)

Find the equation of the plane which cuts the paraboloid $x^2 - 3y^2 = 3z$ in the conic with centre $(1, 2, 3)$. (10)

Since centre of conic lies on the paraboloid
thus plane is locus of parallel chords
bisected at mid point.

Hence equation of plane is given by

$$T = S_1 \quad \text{where } T \rightarrow \text{tangent plane}$$

$S_1 \rightarrow$ centre lies
on paraboloid

given $S: x^2 - 3y^2 = 3z$

centre $(1, 2, 3)$

$$S_1 = 1 - 12 - 9$$

$$S_1 = -20$$

$$T = x(1) - 3y(2) - 3\left(\frac{z+3}{2}\right) = 0$$

$$x - 6y - \frac{3z}{2} - \frac{9}{2} = 0$$

$$T = 2x - 12y - 3z - 9 = 0$$

then plane is $T = S_1$

$$2x - 12y - 3z - 9 = -20$$

$$2x - 12y - 3z + 11 = 0$$

Q.4(a)

Find the equation of the sphere which touches the sphere $x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0$ at the point $(1, 1, -1)$ and passes through the origin. (10)

Given sphere $S: x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0$

Tangent plane to the sphere at point $(1, 1, -1)$

$$x(1) + y(1) + z(-1) - \left(\frac{x+1}{2}\right) + 3\left(\frac{y+1}{2}\right) + z - 1 - 3 = 0$$

$$x + y - z - \frac{x}{2} - \frac{1}{2} + \frac{3y}{2} + \frac{3}{2} + z - 4 = 0$$

$$\frac{x}{2} + \frac{5y}{2} - 3 = 0$$

$$T: x + 5y - 6 = 0$$

thus sphere through circle $(S=0, T=0)$

is

$$S + \lambda T = 0$$

$$(x^2 + y^2 + z^2 - x + 3y + 2z - 3) + \lambda(x + 5y - 6) = 0$$

$$x^2 + y^2 + z^2 + (-1 + \lambda)x + (3 + 5\lambda)y + 2z + (-3 - 6\lambda) = 0 \quad \text{--- (1)}$$

Since it passes through origin

$$-3 - 6\lambda = 0$$

$$\lambda = -\frac{1}{2}$$

thus required equation of sphere is

$$x^2 + y^2 + z^2 - \frac{3}{2}x + \left(3 - \frac{5}{2}\right)y + 2z + (-3 + 3) = 0$$

$$x^2 + y^2 + z^2 - \frac{3x}{2} + \frac{y}{2} + 2z = 0$$

$$2(x^2 + y^2 + z^2) - 3x + y + 4z = 0$$



Q.4(b)

Find the equation of the cone with vertex at $(2a, b, c)$ and passing through the curve $x^2 + y^2 = 4a^2$ and $z = 0$. Find b and c if the cone also passes through the curve $y^2 = 4a(z+a), x=0$. Also show that the cone is cut by the plane $y=0$ in two straight lines and the angle between them is given by $\tan \theta = 2$.

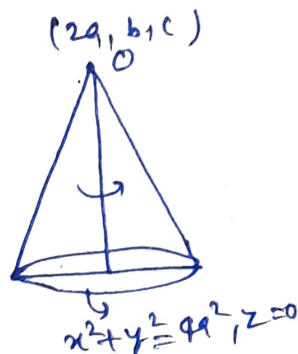
(20)

Given vertex of cone

say $O : (2a, b, c)$

and guiding curve

$$C: x^2 + y^2 = 4a^2, z=0 \quad \text{--- (1)}$$



Equation of axis of cone

$$\frac{x-2a}{1} = \frac{y-b}{m} = \frac{z-c}{n} = t \text{ (say)} \quad \text{--- (2)}$$

where l, m, n are d.c.s of axis

general point on (2)

$$(2a + lt, b + mt, c + nt)$$

for $z=0$

$$\frac{x-2a}{1} = \frac{y-b}{m} = -\frac{c}{n}$$

$$\text{Point } \left(2a - \frac{1c}{n}, b - \frac{mc}{n}, 0 \right)$$

this point lies on $x^2 + y^2 = 4a^2$

$$\therefore \left(2a - \frac{1c}{n} \right)^2 + \left(b - \frac{mc}{n} \right)^2 = 4a^2$$

$$\left(2a - c \left(\frac{x-2a}{z-c} \right) \right)^2 + \left(b - c \left(\frac{y-b}{z-c} \right) \right)^2 = 4a^2$$

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on this margin

$$\left\{ \frac{2a(z-c) - cx + 2ac}{z-c} \right\}^2 + \left\{ \frac{b(z-c) - c(y-b)}{z-c} \right\}^2$$

$$= 4a^2$$

$$\left\{ \frac{2az - 2ac - cx + 2ac}{z-c} \right\}^2 + \left\{ \frac{bz - bc - cy + bc}{z-c} \right\}^2$$

$$= 4a^2$$

$$(2az - cx)^2 + (bz - cy)^2 = 4a^2(z-c)^2 \quad \text{--- (3)}$$

We are given that this cone pass through the curve $y^2 = 4a(z+a)$ --- (4) $x=0$

Put $x=0$ in eqn. (3) $\Rightarrow$

$$4a^2z^2 + b^2z^2 + c^2y^2 - 2bcyz = 4a^2(z^2 + c^2 - 2cz)$$

$$b^2z^2 + c^2y^2 - 2bcyz = 4a^2c^2 - 8a^2cz$$

$$b^2z^2 + c^2y^2 - 2bcyz = 4a^2c(c-2z) \quad \text{--- (5)}$$

Now comparing eqn. (4) + (5) $\Rightarrow$

$$b^2 = 0, \quad \frac{c^2}{1} = \frac{4a}{-8a^2c} - \frac{8a^2c}{4a} = \frac{4a^2c^2}{4a^2}$$

$$\boxed{b=0, c=-2a}$$

Now put $b=0$, $c=-2a$ in equation (3)

$$(2az + 2ax)^2 + 4a^2y^2 = 4a^2(z+2a)^2$$

$$(z+x)^2 + y^2 = (z+2a)^2$$

$$x^2 + y^2 + z^2 + 2xz = z^2 + 4a^2 + 4az$$

$$x^2 + y^2 + 2xz - 4az - 4a^2 = 0$$

Also given that ~~plane~~ cone is cut by
plane $y=0$

Hence $x^2 + 2xz - 4az - 4a^2 = 0$, $y=0$ — (4)

Equation (4) represents two straight lines
angles between them is given by

$$\tan \theta = \frac{2\sqrt{f^2 + g^2 + h^2 - ab - bc - ca}}{a + b + c}$$

$$= \frac{2\sqrt{1 - 1 \times 0 - 0 - 0}}{1 + 0 + 0}$$

$$\boxed{\tan \theta = 2}$$

Q.4(c)

State and prove Rank Nullity theorem for linear transformation.

(20)

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on this marginRank-Nullity theorem ~

Let U and V be vector spaces over the field F and let T be a linear transformation from U into V . If U is finite dimensional then

$$\boxed{\text{rank}(T) + \text{nullity}(T) = \dim U}$$

Proof ~ let dimension of null space of T
i.e. $\dim N(T) = K$

and $\dim U = n$

then we need to prove that $\frac{\text{rank}(T)}{= n - K}$

consider a basis $S = \{\alpha_1, \alpha_2, \dots, \alpha_K\}$ of null space.

Since S is a subset of U and it is linearly independent hence it can be extended to form basis of U .

let $S' = \{\alpha_1, \alpha_2, \dots, \alpha_K, \alpha_{K+1}, \dots, \alpha_n\}$ is a basis of U .

Since vectors $T(\alpha_1), T(\alpha_2), \dots, T(\alpha_n)$ are in range of T , we will

show that $S' = \{T(\alpha_{k+2}), T(\alpha_{k+2}), \dots, T(\alpha_n)\}$

is a basis of range space of T .

(i) first we shall show that S' spans the range.

Let $\beta \in R(T)$, then $\exists \alpha \in U$ such that

$$T(\alpha) = \beta \quad (\because T \text{ is linear transformation})$$

Now $\alpha \in U$ and S' is a basis of U

$$\therefore \text{thus } \alpha = a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n$$

$$\begin{aligned} \therefore T(\alpha) &= T(a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_n \alpha_n) \\ &= a_1 T(\alpha_1) + a_2 T(\alpha_2) + \dots + a_k T(\alpha_k) \\ &\quad + a_{k+2} T(\alpha_{k+2}) + \dots + a_n T(\alpha_n) \end{aligned}$$

($\because T$ is linear transformation)

since $\{\alpha_1, \alpha_2, \dots, \alpha_k\} \in N(T)$

$$\text{hence } T(\alpha_1) = T(\alpha_2) = \dots = T(\alpha_k) = 0$$

$$\therefore T(\alpha) = a_{k+2} T(\alpha_{k+2}) + a_{k+2} T(\alpha_{k+2}) + \dots + a_n T(\alpha_n)$$

$\therefore$ the vectors $T(\alpha_{k+2}), T(\alpha_{k+2}), \dots, T(\alpha_n)$ span the range of T .

(ii) Now we shall show that the vectors
 $T(\alpha_{k+2}), \dots, T(\alpha_n)$ are linearly
 independent.

consider $b_{k+2}, b_{k+2}, \dots, b_n \in F$ such that

$$b_{k+2} T(\alpha_{k+2}) + b_{k+2} T(\alpha_{k+2}) + \dots + b_n T(\alpha_n) = 0$$

$$T(b_{k+2} \alpha_{k+2} + b_{k+2} \alpha_{k+2} + \dots + b_n \alpha_n) = 0$$

{ $\because T$ is L.T }

$$\therefore b_{k+2} \alpha_{k+2} + b_{k+2} \alpha_{k+2} + \dots + b_n \alpha_n \in N(T)$$

thus it can be written in terms of basis
 S .

$$\therefore b_{k+2} \alpha_{k+2} + b_{k+2} \alpha_{k+2} + \dots + b_n \alpha_n = c_1 \alpha_1 + c_2 \alpha_2 + \dots + c_k \alpha_k$$

$$\therefore c_1 \alpha_1 + c_2 \alpha_2 + \dots + c_k \alpha_k - b_{k+2} \alpha_{k+2} - b_{k+2} \alpha_{k+2} - \dots - b_n \alpha_n = 0$$

since $S' = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is set of
 linearly independent vectors.

$$\text{hence } c_1 = c_2 = \dots = b_{k+1} = \dots = b_n = 0$$

thus vectors $T(\alpha_{k+1}), \dots, T(\alpha_n)$ are linearly independent.

thus $S'' = \{T(\alpha_{k+1}), T(\alpha_{k+2}), \dots, T(\alpha_n)\}$ form a basis of range of T .

$$\therefore \text{rank } T = \dim \text{ of } S'' = n - k$$

$$\therefore \boxed{\begin{aligned} \text{rank } T + \text{nullity } T &= n - k + k \\ &= n = \dim U \end{aligned}}$$

Hence Proved

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SECTION-B

Q.5(a)

Find $\lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$

(10)

$$\text{Let } y = \lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$$

$$y = \lim_{x \rightarrow 0} \frac{\frac{1}{x} \log(1+x) e^{\frac{1}{x}} - e}{x}$$

$$= \lim_{x \rightarrow 0} \frac{e^{\frac{1}{x} \left\{ x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \right\}} - e}{x}$$

$$= \lim_{x \rightarrow 0} \frac{e \cdot e^{\left\{ -\frac{x}{2} + \frac{x^2}{3} - \dots \right\}} - e}{x}$$

$$= \lim_{x \rightarrow 0} \frac{e \cdot \left\{ 1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \dots \right) \right\} - e}{x}$$

$$= \lim_{x \rightarrow 0} \frac{-\frac{ex}{2} + \frac{ex^2}{3} + \dots}{x}$$

$$= -\frac{e}{2}$$

Q.5 (b).

Check for the linear dependence the polynomials $i+x+x^2, -(1+i)-2x+2ix^2, x-x^2$ overi. $\mathbb{C}$ ii. $\mathbb{R}$

(10)

(i) for complex field

since

$$-(1+i)-2x+2ix^2 = -(1+i) \{ i+x+x^2 \} + (-1-i) \{ x-x^2 \}$$

Hence $\{ -(1+i)-2x+2ix^2 \}$ can be written in terms of $\{ i+x+x^2, x-x^2 \}$
 thus the given three vectors are linearly dependent in complex field.

(ii) for Real field

$$\text{let } a(i+x+x^2) + b\{ -(1+i)-2x+2ix^2 \} + c\{ x-x^2 \} = 0$$

$$ai - b - bi + x(a-2b+c) + x^2(a+2ib-c) = 0 + 0x + 0x^2$$

comparing $\sim$

$$ai - b - bi = 0 \quad \sim \quad ai = b(1+i)$$

$$a - 2b + c = 0 \quad \sim \quad b\left(\frac{1+i}{i} - 2\right) + c = 0$$

$$a + 2ib - c = 0 \quad \sim \quad b\left(\frac{1-i}{i}\right) + c = 0$$

$$c = -b \left(\frac{1-i}{i} \right)$$

$$\therefore \frac{b(1+i)}{i} + 2i \{ b \} + b \left(\frac{1-i}{i} \right) = 0$$

$$b(1+i+2i^2+1-i) = 0$$

$$b = 0$$

$$a = 0$$

$$c = 0$$

thus given set of vectors are linearly independent in real field.

Q.5 (c)

Integrate the function over the domain $f(x, y) = xy(x^2 + y^2)$ over the domain

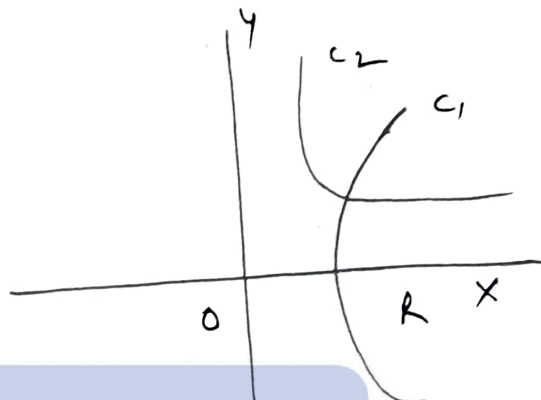
$$R: \{-3 \leq x^2 - y^2 \leq 3, 1 \leq xy \leq 4\}.$$

(10)

consider

$$u = x^2 - y^2$$

$$v = xy$$



to find

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2 + y^2)$$

$$\text{here } |J| = \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{2(x^2 + y^2)}$$

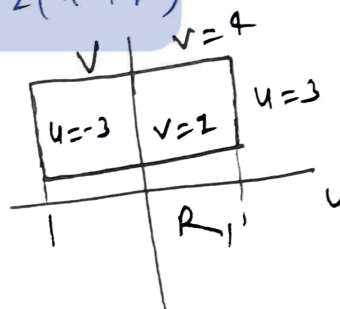
thus

$$\iint_R f(x, y) dx dy$$

$$= \iint_{R_1} f(u, v) |J| du dv$$

$$= \iint_{R_1} xy(x^2 + y^2) \cdot \frac{1}{2(x^2 + y^2)} du dv$$

$$= \int_{v=1}^4 \int_{u=-3}^3 \frac{v}{2} du dv$$



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$$= 2[4]_0^3 \left[\frac{v^2}{4} \right]_1^4$$

$$= 6 \left[\frac{16-1}{4} \right]$$

$$= \frac{3}{2} (15)$$

$$= \frac{45}{2}$$

// Answer



Q.5(d) Calculate approximately $(245)^{\frac{1}{5}}$ using LMV theorem.

(10)

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We know by Lagrange's mean value theorem that

$$f(a+h) = f(a) + h f'(a+\theta h)$$

$$\text{take } a = 243$$

$$h = 2$$

$$f(243+2) = f(243) + 2 f'(a+\theta h)$$

$$\text{where } 0 < \theta < 1$$

$$\text{Consider } f(x) = x^{\frac{1}{5}}$$

$$f'(x) = \frac{1}{5} x^{-\frac{4}{5}}$$

$$\therefore f(245) = 3 + 2 \times \frac{1}{5} (243+2\theta)^{-\frac{4}{5}}$$

$$\text{take } \theta = 0$$

$$f(245) = 3 + 2 \times \frac{1}{5} (3)^{-4}$$

$$= 3 + \frac{2}{5(3)^4}$$

$$\boxed{f(245) = 3.005}$$

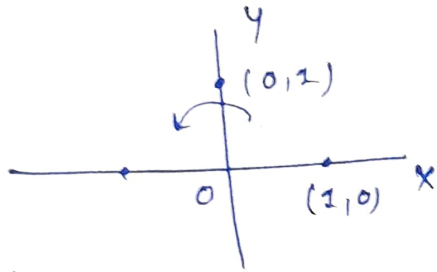
(c)

What 3 by 3 matrix represent the transformation that represent the transformation that rotate the xy plane through 90° leaving the z -axis alone? (10)

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$$T(1, 0, z) = (0, 1, z)$$

$$T(0, 1, z) = (-1, 0, z)$$



where $T: U \rightarrow U$ such that it rotates elements by 90° .

Since $\{(1, 0, z), (0, 1, z), (0, 0, z)\}$ are basis of U .

$$\text{Hence } T(1, 0, z) = (0, 1, z) = 0(1, 0, z) + 1(0, 1, z) + 0(0, 0, z)$$

$$\begin{aligned} T(0, 1, z) &= (-1, 0, z) \\ &= -1(1, 0, z) + 0(0, 1, z) + 0(0, 0, z) \end{aligned}$$

$$T(0, 0, z) = (0, 0, z) = 0(1, 0, z) + 0(0, 1, z) + 1(0, 0, z)$$

Hence Representative matrix is

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

// Answer

Q.6 (a)

A twice differentiable function is such that $f(a) = f(b) = 0$ and $f(c) > 0$, for $a < c < b$. Prove that there is at least one value of ξ between a and b for which $f''(\xi) < 0$. (15)

We are given $f(a) = f(b) = 0$

and $f(c) > 0$ for $a < c < b$

Since f is differentiable in $[a, b]$ it is also continuous in $[a, b]$.

Hence by Lagrange's mean value theorem we know that

$$f'(\eta_1) = \frac{f(c) - f(a)}{c - a} \quad \forall \eta_1 \in [a, c]$$

$$\text{and } f'(\eta_2) = \frac{f(b) - f(c)}{b - c} \quad \forall \eta_2 \in [c, b]$$

Since it is given that f is twice differentiable i.e. $f''(x)$ exists in $[a, b]$

hence $f'(x)$ is continuous and differentiable in $[a, b]$

Again applying Lagrange's mean value theorem

$$f''(\xi) = \frac{f'(\eta_2) - f'(\eta_1)}{\eta_2 - \eta_1} \quad \forall \eta_1, \eta_2 \in [a, b]$$

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$$f''(\xi) = \frac{\frac{f(b)-f(c)}{b-c} - \frac{f(c)-f(a)}{c-a}}{h_2-h_1}$$

Since $f(c) > 0$ {given}
and $f(a) = f(b) = 0$

$$\begin{aligned} f''(\xi) &= \frac{-\frac{f(c)}{b-c} - \frac{f(c)}{c-a}}{h_2-h_1} \\ &= \frac{-f(c) \left[\frac{c-a+b-c}{(b-c)(c-a)} \right]}{h_2-h_1} \\ &= -f(c) \frac{(b-a)}{(b-c)(c-a)} \cdot \frac{1}{h_2-h_1} \end{aligned}$$

Since $f(c) > 0$
 ~~$a < b < c$~~ $a < c < b$
and $h_2 > h_1$

$$\boxed{f''(\xi) < 0} \quad \text{for some } \xi \in [a, b]$$

Hence proved

Q.6 (b)

Find the maximum and minimum values of $f(x,y) = xy - x^3y^2$ when (x,y) runs over the square $0 \leq x \leq 1, 0 \leq y \leq 1$. (20)

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$$\text{Given } f(x,y) = xy - x^3y^2$$

$$f_x = y - 3x^2y^2$$

$$f_y = x - 2x^3y$$

$$\text{Put } f_x = f_y = 0$$

$$y - 3x^2y^2 = 0$$

$$y(1 - 3x^2y) = 0$$

$$y = 0 \text{ or } 3x^2y = 1$$

$$\text{If } 3x^2y = 1 = 2x^2y$$

$$3 = 2 \text{ (Not possible)}$$

Here $(0,0)$ is an extreme point

$$r = f_{xx} = -6xy^2$$

$$s = f_{xy} = 1 - 6x^2y$$

$$t = f_{yy} = -2x^3$$

$$\text{at } (0,0) \quad \Delta = f(0,0) = 0$$

Now Boundary

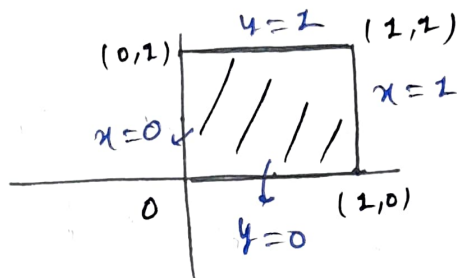
$$0 \leq x < 1, \quad 0 \leq y \leq 1$$

(i) where $y=0$

$$f(x,0)=0$$

(ii) where $x=0$

$$f=0$$

(iii) where $x=1$

$$f(1,y) = y - y^2$$

$$f_y = 1 - 2y = 0 \quad \& \quad f_{yy} = -2 < 0 \quad (\text{max})$$

$$y = 1/2$$

critical point $(1, 1/2)$

$$f(1, 1/2) = 1\left(\frac{1}{2}\right) - 1^3\left(\frac{1}{2}\right)^2$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

(iv) where $y=1$

$$f(x,1) = x - x^3$$

$$f_x = 1 - 3x^2 = 0$$

$$x = \pm \frac{1}{\sqrt{3}}$$

critical points $\left(\frac{1}{\sqrt{3}}, 1\right) + \left(-\frac{1}{\sqrt{3}}, 1\right)$

$$f_{xx} = -6x$$

$$f_{xx}\left(\frac{1}{\sqrt{3}}, 1\right) = -ve < 0 \quad (\text{max})$$

Here $\left(-\frac{1}{\sqrt{3}}, 1\right)$ is outside the given region

$$\text{Hence } f\left(\frac{1}{\sqrt{3}}, 1\right) = \frac{1}{\sqrt{3}}(1) - \left(\frac{1}{\sqrt{3}}\right)^3 \cdot 1$$

$$= \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \left(\frac{2}{3}\right)$$

$$= \frac{2}{3\sqrt{3}}$$

Thus All critical point and values of f
at those point

Point	$(0, 0)$	$(1, 0)$	$(0, 1)$	$(1, 1)$	$(1, 1/2)$	$(\frac{1}{\sqrt{3}}, 1)$
f	0	0	0	0	$1/4$	$\frac{2}{3\sqrt{3}}$

minima

maxima

Q.6 (c)

Prove that the infinite subset $S = \{1, x, x^2, \dots\}$, of the vector space P of all real polynomials in x , is linearly independent. (15)

We need to prove that $S = \{1, x, x^2, \dots\}$ is linearly independent.

consider $c_1, c_2, c_3, \dots, c_n, \dots \in F$

such that "

$$c_1(1) + c_2x + \dots + c_nx^n + \dots = 0$$

$$c_1(1) + c_2x + \dots + c_nx^n + \dots = 0 + 0x + 0x^2 + \dots + 0x^n + \dots$$

Comparing "

$$c_1 = c_2 = \dots = c_n = \dots = 0$$

Hence S is linearly independent.

Since $S = \{1, x, x^2, \dots, x^n, \dots\}$

Here every vector ~~is~~ has power greater than preceding vectors

hence it's not possible to write a vector as combination of other

vectors. Thus S is linearly

independent.

7 (a) Show that the function $f(x, y)$ is differentiable at the origin

$$f(x, y) = \begin{cases} xy \cdot \frac{x^2 - y^2}{x^2 + y^2}, & \text{if } x^2 + y^2 \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases} \quad (15)$$

We know by definition of differentiability

$$f(h, k) - f(0, 0) = f_x(0, 0) \cdot h + f_y(0, 0) \cdot k + h \phi(h, k) + k \psi(h, k)$$

such that $\phi(h, k)$ & $\psi(h, k)$ tends to zero as $(h, k) \rightarrow (0, 0)$

thus finding partial derivatives at origin

$$\begin{aligned} f_x(0, 0) &= \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0 \end{aligned}$$

$$\begin{aligned} f_y(0, 0) &= \lim_{k \rightarrow 0} \frac{f(0, k) - f(0, 0)}{k} \\ &= \lim_{k \rightarrow 0} \frac{0 - 0}{k} = 0 \end{aligned}$$

$$\therefore f(h, k) - f(0, 0) = h \phi(h, k) + k \psi(h, k)$$

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$$hk \frac{h^2 - k^2}{h^2 + k^2} - 0 = h\phi(h, k) + k\psi(h, k)$$

$$h \frac{kh^2}{h^2 + k^2} - k \frac{hk^2}{h^2 + k^2} = h\phi(h, k) + k\psi(h, k)$$

Comparing,

$$\phi(h, k) = \frac{kh^2}{h^2 + k^2} \quad \& \quad \psi(h, k) = \frac{-hk^2}{h^2 + k^2}$$

$$\text{At } (h, k) \rightarrow (0, 0) \quad \phi(h, k) = 0 \quad \& \quad \text{At } (h, k) \rightarrow (0, 0) \quad \psi(h, k) = 0$$

thus condition for differentiability is satisfied.

Hence $f(x, y)$ is differentiable at
origin.

Q.7 (b)

Show that

$$\int_0^{\pi/2} \log(1 - x^2 \sin^2 \theta) d\theta = \pi \log(1 + \sqrt{1 - x^2}) - \pi \log 2, \text{ if } |x| < 1. \quad (15)$$

$$\text{Let } I = \int_0^{\pi/2} \log(1 - x^2 \sin^2 \theta) d\theta$$

Differentiating w.r. to x

$$\frac{dI}{dx} = \int_0^{\pi/2} \frac{1}{1 - x^2 \sin^2 \theta} \cdot -2x \sin^2 \theta d\theta$$

$$= -2x \int_0^{\pi/2} \frac{\sin^2 \theta}{1 - x^2 \sin^2 \theta} d\theta$$

$$= -\frac{2}{x} \int_0^{\pi/2} \frac{\sin^2 \theta}{\frac{1}{x^2} - \sin^2 \theta} d\theta$$

$$= -\frac{2}{x} \int_0^{\pi/2} \frac{x^2 \sin^2 \theta}{1 - x^2 \sin^2 \theta} d\theta$$

$$= +\frac{2}{x} \int_0^{\pi/2} \frac{1 - x^2 \sin^2 \theta - 1}{1 - x^2 \sin^2 \theta} d\theta$$

$$= \frac{2}{x} \int_0^{\pi/2} d\theta - \frac{2}{x} \int_0^{\pi/2} \frac{d\theta}{1 - x^2 \sin^2 \theta}$$

$$= \frac{2}{x} \cdot [\theta]_0^{\pi/2} - \frac{2}{x} \int_0^{\pi/2} \frac{\operatorname{cosec}^2 \theta d\theta}{\operatorname{cosec}^2 \theta - x^2}$$

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$$\frac{2}{x} \cdot \left(\frac{\pi}{2} - 0 \right) - \frac{2}{x} \int_0^{\pi/2} \frac{\operatorname{cosec}^2 \theta d\theta}{\cot^2 \theta + 2 - x^2} \quad \text{--- (1)}$$

$$\text{Let } \cot \theta = t$$

$$-\operatorname{cosec}^2 \theta d\theta = dt$$

$$\text{Limit } t = \infty \text{ when } \theta = 0$$

$$t = 0 \text{ when } \theta = \pi/2$$

Here from (1),

$$= \frac{\pi}{x} - \frac{2}{x} \int_{\infty}^0 \frac{-dt}{(1-x^2)+t^2}$$

$$= \frac{\pi}{x} - \frac{2}{x} \int_0^{\infty} \frac{dt}{(\sqrt{1-x^2})^2 + t^2}$$

$$= \frac{\pi}{x} - \frac{2}{x} \left[\tan^{-1} \left(\frac{t}{\sqrt{1-x^2}} \right) \right]_0^{\infty} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{\pi}{x} - \frac{2}{x} \cdot \frac{1}{\sqrt{1-x^2}} \left[\tan^{-1}(\infty) - \tan^{-1}(0) \right]$$

$$= \frac{\pi}{x} - \frac{2}{x\sqrt{1-x^2}} \cdot \frac{\pi}{2}$$

$$\frac{dI}{dx} = \frac{\pi}{x} \left[1 - \frac{1}{\sqrt{1-x^2}} \right]$$

$$= \frac{\pi}{x} - \pi \left[\frac{1-x^2+x^2}{x\sqrt{1-x^2}} \right]$$

$$= \frac{\pi}{x} - \pi \frac{\sqrt{1-x^2}}{x} - \pi \frac{x}{\sqrt{1-x^2}}$$

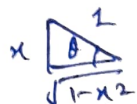
$$\frac{dI}{dx} = \frac{\pi}{x} - \frac{\pi}{x\sqrt{1-x^2}}$$

integrating with respect to x ,

$$I = \pi \log x - \int \frac{\pi}{x\sqrt{1-x^2}} dx + C \quad (2)$$

$$\text{let } x = \sin \theta$$

$$dx = \cos \theta d\theta$$



$$\text{then } \int \frac{dx}{x\sqrt{1-x^2}} = \int \frac{\cos \theta d\theta}{\sin \theta \cdot \cos \theta}$$

$$= \int \cot \theta d\theta$$

$$= \log (\cot \theta)$$

$$= \log \left(\frac{1}{x} \cdot \frac{\sqrt{1-x^2}}{x} \right)$$

put in equation (2),

$$I = \pi \log x - \pi \log \left(\frac{\sqrt{1-x^2}}{x} \right) + C$$

~~Now we know that $I(0) = 0$~~

$$I = \pi \log \left(\frac{x^2}{\sqrt{1-x^2}} \right) + C$$

$$I = \pi \log \left(\frac{x^2(1+\sqrt{1-x^2})}{1-1+x^2} \right) + C$$

$$I = \pi \log (1+\sqrt{1-x^2}) + C$$

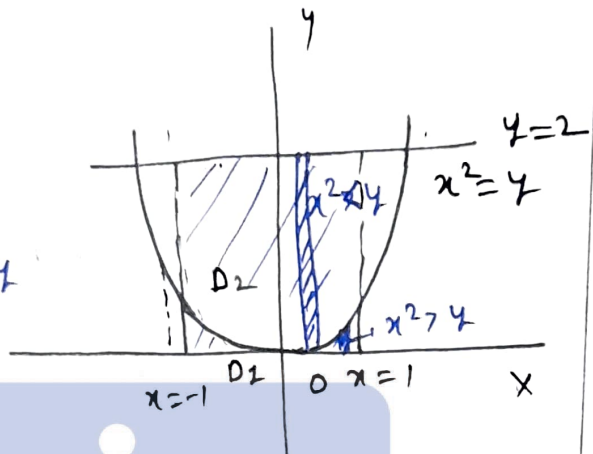
$$\text{Since } I(0) = 0 \quad \therefore C = -\pi \log 2$$

$$\therefore \boxed{I = \pi \log (1+\sqrt{1-x^2}) - \pi \log 2}$$

Q.7 (c) Evaluate $\iint_R \sqrt{|y-x^2|} dx dy$ where $R = [-1, 1; 0, 2]$.

(20)

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$$\begin{aligned}
 & \iint_R \sqrt{|y-x^2|} dx dy \\
 &= \iint_{D_1} \sqrt{|y-x^2|} dx dy + \iint_{D_2} \sqrt{|y-x^2|} dx dy \\
 &= \int_{-1}^1 \int_0^{x^2} \sqrt{x^2-y} dx dy + \int_{-1}^1 \int_{x^2}^2 \sqrt{y-x^2} dx dy \\
 &= \int_{-1}^1 \left[\frac{(x^2-y)^{3/2}}{3/2} \right]_0^{x^2} dx + \int_{-1}^1 \left[\frac{(y-x^2)^{3/2}}{3/2} \right]_{x^2}^2 dx \\
 &= \int_{-1}^1 \frac{2}{3} |x|^3 dx + \frac{2}{3} \int_{-1}^1 (2-x^2)^{3/2} dx \\
 &= \int_{-1}^0 \frac{2}{3} (-x)^3 dx + \int_0^1 \frac{2}{3} (x)^3 dx + \frac{2}{3} \cdot 2 \int_0^1 (2-x^2)^{3/2} dx \\
 &= -\frac{2}{3} \left[\frac{x^4}{4} \right]_{-1}^0 + \frac{2}{3} \left[\frac{x^4}{4} \right]_0^1 + \frac{4}{3} \int_0^1 (2-x^2)^{3/2} dx
 \end{aligned}$$


$$-\frac{2}{3}\left(-\frac{1}{4}\right) + \frac{2}{3}\left(\frac{1}{4}\right) + \frac{4}{3} \int_0^1 (2-x^2)^{3/2} dx \quad \text{--- ①}$$

$$\text{let } x = \sqrt{2} \sin \theta$$

$$dx = \sqrt{2} \cos \theta$$

$$\text{limit } \theta = 0 \text{ when } x = 0$$

$$\theta = \pi/4 \text{ when } x = 1$$

$$\begin{aligned} \text{then } \int_0^1 (2-x^2)^{3/2} dx &= \int_0^{\pi/4} 2\sqrt{2} \cos^3 \theta \cdot \sqrt{2} \cos \theta d\theta \\ &= 4 \int_0^{\pi/4} \cos^4 \theta d\theta \\ &= 4 \int_0^{\pi/4} \left(\frac{1 + \cos 2\theta}{2} \right)^2 d\theta \\ &= \int_0^{\pi/4} (1 + \cos^2 2\theta + 2 \cos 2\theta) d\theta \end{aligned}$$

$$= \int_0^{\pi/4} \left(1 + \frac{\cos 4\theta + 1}{2} + 2 \cos 2\theta \right) d\theta$$

$$= \left[\theta + \frac{\sin 4\theta}{4} + \frac{\theta}{2} + 2 \frac{\sin 2\theta}{2} \right]_0^{\pi/4}$$

$$= \frac{\pi}{4} \cdot \frac{3}{2} + 1 = \frac{3\pi}{8} + 1$$

put in equation ① $\rightarrow$

$$= \frac{1}{3} + \frac{4}{3} \left(1 + \frac{3\pi}{8} \right)$$

$$= \frac{1}{3} + \frac{\pi}{2} + \frac{4}{3}$$

$$\boxed{I = \frac{5}{3} + \frac{\pi}{2}}$$

// Answer