NEXT IAS

(To be filled by candidate)

Name of Candidate : BHAGWATA SAIKIA

Roll No. : .....

Registration Number : ..... Date of Examination : 10.09.2023

Exam Centre : Old Rajinder Nagar ☐ Bhopal ☐ Online ☐

CSE (MAINS) TEST SERIES - 2023

Test-08	MATHEMATICS OPTIONAL
Dated : 27-08-2023	*Full Syllabus (Paper-II)*

Time Allowed : Three Hours | Maximum Marks: 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

*There are **EIGHT** questions divided in **Two sections** and printed in **ENGLISH**.*

*Candidate has to attempt **FIVE** questions in all.*

***Question No. 1 and 5** are compulsory and out of the remaining, any **THREE** are to be attempted choosing at least **ONE** question from each section.*

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission. Certificate which must be stated clearly on the cover of this Question-cum- Answer (QCA). Booklet in the space provided.

No marks will be given for answers written in a medium other than the authorized one.

Word limit in questions, wherever specified, should be adhered to.

Attempts of question shall be counted in sequential order. Unless struck off. Attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question cum Answer (QCA). Booklet must be clearly struck off.

DELHI CENTRE:
27-B, Pusa Road, Metro Pillar no. 118,
Near Karol Bagh Metro, New Delhi-110060
Phone: 8081300200

BHOPAL CENTRE:
Plot No. 46 Zone - 2, M.P Nagar,
Bhopal - 462011
Phone: 8827664612, 8081300200

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Q. No.	Page No.	Max. Marks	Marks	Total	Signature
1. (a)	4	10			
1. (b)	6	10			
1. (c)	8	10			
1. (d)	10	10			
1. (e)	12	10			
2. (a)	14	15			
2. (b)	17	15			
2. (c)	20	20			
3. (a)	24	15			
3. (b)	27	20			
3. (c)	31	15			
4. (a)	34	15			
4. (b)	37	20			
4. (c)	41	15			
5. (a)	44	10			
5. (b)	46	10			
5. (c)	48	10			
5. (d)	50	10			
5. (e)	52	10			
6. (a)	54	20			
6. (b)	58	15			
6. (c)	61	15			
7. (a)	64	20			
7. (b)	68	15			
7. (c)	71	15			
8. (a)	74	15			
8. (b)	77	15			
8. (c)	80	20			
			Grand Total		

Remarks

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SECTION-A

1. (a) Prove that every Maximal Ideal of a commutative ring with identity is a prime ideal. (10)

Let M be a maximal ideal of commutative ring with identity R.

Hence, R/M is a field.

$\Rightarrow$ R/M is an integral domain

$\Rightarrow$ M is a prime ideal.

Hence proved

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3. (b) If $f(z)$ is integral function, satisfying the inequality $|f(z)| \leq M$ for all values of z, where M is a positive constant, then $f(z)$ is constant. (10)

Let $f(z) = u + iv$ Given u, v functions of z

Given $u^2 + v^2 \leq M$.

$$|f'(z_0)| = \left|\frac{1}{2\pi i}\oint \frac{f(z)\,dz}{(z-z_0)^2}\right| \leq \frac{1}{2\pi}\oint \left|\frac{f(z)}{(z-z_0)^2}\right| dz \quad \text{--- (1)}$$

Take $z_0 = z + re^{i\theta}$

taking $|z - z_0| = r$

In (1)

$$\frac{1}{2\pi}\oint \frac{|f(z)|}{|(z-z_0)^2|}\,dz \leq \frac{1}{2\pi}\oint \frac{M}{r^2}\,dz$$

$$\leq \frac{1}{2\pi}\cdot\frac{M}{r^2} \times 2\pi r$$

$$\leq \frac{M}{r}$$

Now, taking $r \to \infty$

we have

$$|f'(z_0)| \leq 0 \Rightarrow f'(z_0) = 0$$

As, z_0 is arbitrary, $f(z)$ is constant

Hence Proved

1. (c) Show that the sequence $\langle f_n \rangle$ where $f_n(x) = \tan^{-1} nx$ is not uniformly convergent on $[0,1]$. (10)

pointwise limit for at any $x \neq 0$ is

$$l = \lim_{n \to \infty} \tan^{-1} nx = \tan^{-1} \infty = 0$$

at $x = 0$, $l = \tan^{-1} 0 = 0$

Hence pointwise limit is 0.

Now, To be uniformly convergent for any $\epsilon > 0$, $\forall x$, $\exists N$ such that for $n \geq N$

$$|f_n(x) - l| < \epsilon \quad \text{—(1)}$$

If we take $\epsilon = \frac{\pi}{8}$, and $x = \frac{1}{n}$ for any n

$f_n(x)$ becomes $\tan^{-1} n \cdot \frac{1}{n}$

$= \tan^{-1} 1$

$= \frac{\pi}{4}$

at $x = 1/n$

Here $|f_n(x) - l| = \left|\frac{\pi}{4} - 0\right| > \frac{\pi}{8} > \epsilon$

Hence not uniformly convergent

1. (d) Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

$\sum_{n=1}^{\infty} a_n$ Converges

let- $\lim_{M\to\infty} \sum_{n=1}^{M} a_n = l$

Now, $a_n = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_i$

taking $n \to \infty$

$= \lim_{n\to\infty} \sum_{i=1}^{n} a_i - \lim_{n\to\infty} \sum_{i=1}^{n-1} a_i$

$= l - l$

$= 0$

Hence proved

1. (e) Give an example of a ring having identity but a subring of this having a different identity. (10)

Example of such ring = $\{M_{2\times 2}\}$ = {set of all 2×2 real matrices}

And a subring of this is

$$S = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{R} \right\}$$

Here the identity will be

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

because

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$$

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2. (a) Derive Cauchy's Integral Formula.

Cauchy's integral formula is

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,dz$$

or $$2\pi i f(a) = \oint \frac{f(z)}{z-a}\,dz$$

(where, $z = a + re^{i\theta}$

So, $dz = ire^{i\theta}\,d\theta$

$$\frac{dz}{iz} = d\theta$$

So, $$\oint \frac{f(z)}{z-a} = \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\,d\theta \quad -(1)$$

Now by taylor series approximation (as r can be shrinked to as low as possible (deformation of contour)

$$f(a+re^{i\theta}) = f(a) + f'(a)re^{i\theta} \text{ and higher order term}$$

So, (1)

$$\Rightarrow i\oint (f(a) + f'(a)re^{i\theta})\,d\theta$$

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$$= i\left[f(a) \times 2\pi\right] + i \int_0^{2\pi} f'(a)\, r\, e^{i\theta}\, d\theta$$

$$= f(a)\, 2\pi i + f'(a)\, i \int_0^{2\pi} e^{i\theta}\, d\theta$$

$$= 2\pi i f(a)$$

Hence proved

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2. (b) If $f:[a,b]\to\mathbb{R}$ is monotone on $[a,b]$, then $f\in R[a,b]$. (15)

Proof Without loss of generality, let us consider a uniform partition. monotonic increasing

Let us consider a partition

$P=\{\underset{x_0}{a}, x_1, x_2, \ldots, x_n=b\}$ where $x_i = x_0 + i\times h$

Now, lower sum $L(P) = f(x_0)(x_1-x_0) + f(x_1)(x_2-x_1) + \ldots + f(x_{n-1})(x_n-x_{n-1})$

$= f(x_0)h + f(x_1)h + \ldots f(x_{n-1})h$ — (1)

upper sum $U(P) = f(x_1)h + f(x_2)h + \ldots + f(x_n)h$ — (2)

So $|U(P)-L(P)| = |f(x_n)h - f(x_0)h|$

$= h|f(b)-f(a)|$ — (3)

For given $\epsilon>0$

we can take $h = \dfrac{\epsilon}{2|f(b)-f(a)|}$

So, that $|U(P)-L(P)| = h|f(b)-f(a)|$ from (3)

$= \frac{\epsilon}{2}|f(b)-f(a)| < \epsilon$

Hence proved

2. (c) If F is a finite field, prove that it's characteristic must be a prime number 'p' and F contains p^n elements for some integer 'n'. Further PT if $a \in F$ then $a^{p^n} = a$. (20)

Let the characteristic be an composite number mn. $(m, n > 1)$

So, $\underbrace{1 + (1 + 1 + \cdots + 1}_{mn\text{ times}} = 0$

$$\underbrace{(1 + 1 + \cdots 1}_{m})\underbrace{(1 + 1 + \cdots}_{n}) = 0$$

As field is a integral domain,

$$\underbrace{1 + (1 + 1 + 1 \cdots}_{m} = 0$$

& $\underbrace{1 + (1 + 1 + \cdots}_{n} = 0$

As, $m, n < mn$

hence contradiction, as by definition of characteristic mn is the minimum number for which

$$\underbrace{1 + (1 + \cdots}_{mn} = 0$$

So, characteristic must be prime

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No of elements in field (let is be x)

as the elements form an abelian group so for any element a.1

$p \cdot 1_a = 0$

$\therefore p/x$

If any other number divides x, then.

$x \cdot 1 = 0$

$c(p \cdot 1 = 0$

$c(p \cdot 1) = 0$

$\therefore p/q$

Here, size is p^n elements

Now $a \in F$ we know, $F^* = F - \{0\}$ is product group

here $a^{p^n-1} = e$

$\Rightarrow a^{p^n-1} \cdot a = e \cdot a$

$\Rightarrow a^{p^n} = a$

then proved

3. (a) Prove that $R[x]$ is an integral domain iff R is an integral domain. (15)

Let R be an integral domain.

To prove R[x] integral domain.

Let a(x), b(x) two ~~polynomial~~ non zero polynomial in R[x] of degree m & n respectively.

So, $a(x) = a_0 x + a_1 x + \cdots a_m x^m \quad a_m \neq 0$

$b(x) = b_0 + b_1 x + \cdots + b_n x^n \quad b_n \neq 0$

$a(x)\,b(x) = a_0 b_0 + \cdots + a_m b_n x^{m+n}$

Now, as R is integral domain.

& $a_m \neq 0$, & $b_n \neq 0$.

So, $a_m b_n \neq 0$

hence $a(x)b(x) \neq 0$

hence R[x] is integral domain.

Conversely

If R[x] is ~~not~~ integral domain, any two non zero

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element product is non zero.

Any element of R can be interpreted as constant term of R[x]

hence two non-zero element of R will also give non zero product

Hence, R is integral domain

Hence proved

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3. (b) For a company engaged in manufacturing of three products X, Y and Z, the available data are given below. Determine the product mix to maximise the profit

Operations	Time in hrs required per unit of			Total available hrs per month
	X	Y	Z	
1	1	2	2	200
2	2	1	1	220
3	3	1	2	180
Profit/unit	10	15	8	
Minimum sales requirements per months in units	10	20	30	

(20)

Let they produce x unit of X, y unit of Y, and z unit of Z

So, $x + 2y + 2z \leq 200$ — (1)

$2x + y + z \leq 220$ — (2)

$3x + y + 2z \leq 180$ — (3)

$x \geq 10$ — (4)

$y \geq 20$ — (5)

$z \geq 30$ — (6)

~~And~~ by (2) & (3), it can be seen that 2 is automatically satisfied if (3) is satisfied

We need to maximize.

$10x + 15y + 8z$

effectively ①, ②, ③ can be reduced to

$x + 2y + 2z \le 200$ — (2')

& $3x + y + 2z \le 180$ — (8')

by using x' for $x - 10$, $y' = y - 20$,

& $z' = z - 30$, the constraints becomes

$x' + 10 + 2y' + 40 + 2z' + 60 \le 200$

$= x' + 2y' + 2z' \le 90$ — (9)

& $3x' + y' + 2z' \le 70$ — (10)

$x', y', z' \ge 0$

---

let optimum value of $z' = n$

So, $x' + 2y' \le 90 - 2n$

$3x' + y' \le 70 - 2n$ } equality for optimal

So, $3x' + y' = x' + 2y'$ — (11)

$\Rightarrow 2x' = y'$

$y' = 4x'$ (11')

So, $x' + 4x' = 90 - 2z'$

$\Rightarrow 5x' + 2z' = 90$

As, coeff of x' is more than coeff of z'

we need to optimize x' keeping z' as low as possible.

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Hence, $x' = 18,\ z' = 20,\ y' = 36$

$\therefore x = 28,\ y = 56,\ z = 30$

is the required product-mix for max profit

Maximize $10x' + 15y' + 8z'$

$x' + 2y' + 2z' + s_1 + 0s_2 \leq 90$

$3x' + y' + 2z' + 0s_1 + s_2 = 70$

$x', y', z', s_1, s_2 \geq 0$

	c_j	10	15	8	0	0		
		x'	y'	z'	s_1	s_2	b	θ
0	s_1	1	[2]	2	1	0	90	45 →
0	s_2	3	1	2	0	1	70	70
	$c_j - z_j$ =	10	15 ↑	8	0	0		

15	y'	$\frac{1}{2}$	1	1	$\frac{1}{2}$	0	45	90
0	s_2	[5/2]	0	1	$-\frac{1}{2}$	1	25	10 →
	$c_j - z_j$ =	5/2 ↑	0	−7	$-\frac{15}{2}$	0		

15	y'	0	1	4/5	3/5	−1/5	40
10	x'	1	0	2/5	−1/5	2/5	10
$c_j - z_j$		0	0	−8	−7	−1	

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$x' = 10, y' = 40, z' = 0$

So, phase values are

$\boxed{20, 60, 30}$

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3. (c) Find analytic function $f(z) = u(r,\theta) + iv(r,\theta)$ such that $v(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2$. (15)

Now, $\frac{\partial u}{\partial r} = \frac{\partial v}{r\partial\theta}$ — (i)

$\frac{\partial v}{\partial\theta} = \frac{1}{r}\left(r^2(-\sin 2\theta)2 - r(-\sin\theta)\right)$

$= -2r\sin 2\theta + \sin\theta$

So, $\frac{\partial u}{\partial r} = -2r\sin 2\theta + \sin\theta$

$\Rightarrow u = -\frac{2r^2}{2}\sin 2\theta + r\sin\theta + f(\theta)$ — (iii)

Now, $\frac{\partial u}{r\partial\theta} = -\frac{\partial v}{\partial r}$

$\Rightarrow \frac{\partial u}{r\partial\theta} = -(2r\cos 2\theta - \cos\theta)$

$\Rightarrow \frac{\partial u}{\partial\theta} = -(2r^2\cos 2\theta - r\cos\theta)$

$\Rightarrow u = -\left(\frac{2r^2}{2}\sin 2\theta - r\sin\theta\right) + f(r)$ — (iv)

By (iii) & (iv)

$u = -r^2\sin 2\theta + r\sin\theta + c$

Now, $f'(z) = \frac{du}{dr} + i\frac{dv}{dr}$

$= 1 -$

$$f(z) = u + iv$$

$$= (-r^2 \sin 2\theta + r\sin\theta + c) + i(r^2\cos 2\theta - r\cos\theta + 2)$$

$$= i(r^2(\cos 2\theta + i\sin 2\theta)) - i(r(\cos\theta + i\sin\theta)) + (c + 2i)$$

$$= iz^2 - iz + k$$

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SECTION-B

5. (a) Find the complete integral of the equation $(p+q)(z-xp-yq)=1$. (10)

$$(p+q)(z-xp-yq)=1$$

$$\Rightarrow z-xp-yq=\frac{1}{p+q}$$

$$\Rightarrow z=xp+yq+\frac{1}{p+q}$$

This is in Clairaut form

So the required solution is

$$z=ax+by+\frac{1}{a+b}$$

putting $b=f(a)$

$$z=ax+f(a)y+\frac{1}{a+f(a)}$$

5. (b) Write down the algorithm for solving the equation: $xe^x - 1 = 0$ by bisection method, correct to 4 decimal places. (10)

1. define error $\epsilon = 0.00005$
2. define $f(x) = xe^x - 1$
3. define $f'(x) = e^x + xe^x$
4. define $x = 1$ (initial value)
5. define $Err = 1$
6. ~~define~~ while $(Err > \epsilon)$ {
   $y = x$
   $x = x - \frac{f(x)}{f'(x)}$
   $Err = |x - y|$
   }
7. Output x

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Give a Boolean Expression for the following statements:

i. Y is 1 only when A is 1 and B is 1 or if A is zero and B is 0

ii. Y is 1 only if A, B and C are all 1s or if only one of the variables is 0. (10)

1. $Y = AB + \bar{A}\bar{B}$

$\overline{\bar{A}B + \bar{B}A}$

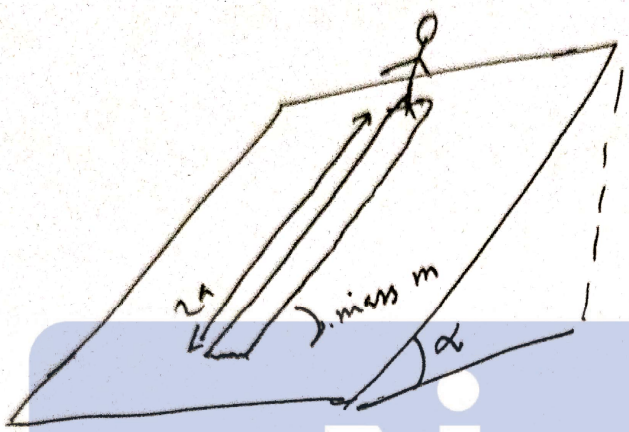
$(A+\bar{B})(A+\bar{A})$

$AB + \bar{B}\bar{A}$

2. $Y = ABC + \bar{A}BC + A\bar{B}C + AB\bar{C}$

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5. (d) A plank of mass m and length $2a$ is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon, a man of mass, M, starting from the upper end walks down the plank so that it does not move, show that he will reach the other end in time $\sqrt{\dfrac{4Ma}{(m+M)g\sin\alpha}}$. (10)

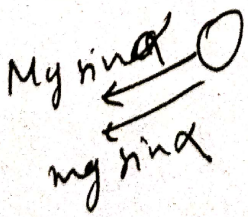


free body diagram



The downward force on plank must be balanced by the man by same upward force of mg sinα

Now, force on man in direction of the slope



here acc^{n} $_2$ $\dfrac{(M+m)}{M}g\sin\alpha$

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using

$$s = ut + \frac{1}{2}at^2$$

$$2a = \frac{1}{2}\,\frac{M+m}{M}\,g\sin\alpha\, t^2$$

$$\sqrt{\frac{4aM}{(M+m)g\sin\alpha}} = t$$

Hence proved

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5. (c) (i) Give an example of a flow for which the stream function exist but velocity potential does not exist.

(ii) Describe the flow for the complex potential given by $W(z) = \frac{k}{\pi}\tan^{-1}\left(\frac{z}{c}\right)$. (10)

(i) velocity potential exists only for irrotational flow.

∴ So, for 2D rotational flow stream function exist but velocity potential doesn't exist

(ii) $W = \frac{k}{\pi}\tan^{-1}\left(\frac{z}{c}\right)$

$|q| = \left|\frac{dW}{dz}\right|$

$= \frac{k}{\pi}\cdot\frac{\frac{1}{c}}{1+\frac{z^2}{c^2}}$

$= \left|\frac{k}{\pi}\cdot\frac{c}{c^2+z^2}\right|$

$= \left|\frac{kc}{\pi}\cdot\frac{1}{c^2+x^2-y^2+2xyi}\right|$

$= \left|\frac{kc}{\pi}\cdot\frac{(c^2+x^2-y^2)-(2xy)i}{(c^2+x^2-y^2)^2+(2xy)^2}\right|$

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$$= \left(\frac{kc}{\pi}\right) \sqrt{\frac{(c^2+x^2-y^2)^2}{((c^2+x^2-y^2)^2+4x^2y^2)^{\frac{1}{2}}} \frac{4x^2y^2}{(c^2+x^2-y^2)+4x^2y^2}}$$

$$= \frac{kc}{\pi} \sqrt{\frac{1}{(c^2+x^2-y^2)^2+4x^2y^2}}$$

$$= \frac{kc}{\pi} \sqrt{\frac{1}{c^4+x^4+y^4+2c^2x^2-2c^2y^2+2x^2y^2}}$$

$$= \frac{kc}{\pi} \sqrt{\frac{1}{(c^2+x^2+y^2)^2-4c^2y^2}} \quad -(1)$$

So, flow is undefined when

$c^2+x^2+y^2 = 2cy$ also at $c^2+x^2+y^2 = -2cy$

$\Rightarrow (c-y)^2+x^2 = 0$ $\Rightarrow (c+y)^2+x^2 = 0$

at, $(0, c)$ $(0, -c)$

For eqn (1) $\Rightarrow$

The magnitude of velocity is symmetrical about x & y axes.

8. (a) Reduce the following partial differential equation to canonical form and hence solve it: $yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0$. (15)

Comparing with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

$R = y,\ S = x+y,\ T = x$

Now, $S^2 - 4RT$

$= x^2 + y^2 + 2xy - 4xy$

$= (x-y)^2 > 0$

hence the equation is hyperbolic

Now solving.

$$R\lambda^2 + S\lambda + T = 0$$

$$y\lambda^2 + (x+y)\lambda + x = 0$$

$$\lambda = \frac{-(x+y) \pm \sqrt{(x+y)^2 - 4xy}}{2y}$$

$$= \frac{-(x+y) \pm (x-y)}{2y}$$

$$= \frac{-x-y+x-y}{2y},\ \frac{-x-y-x+y}{2y}$$

$$= -1,\ -\frac{x}{y}$$

Putting in $\frac{dy}{dx} + \lambda = 0$

$\Rightarrow \frac{dy}{dx} - 1 = 0$ & $\frac{dy}{dx} - \frac{x}{y} = 0$

$\Rightarrow y - x = c_1$ $\Rightarrow y\,dy = x\,dx$

$\Rightarrow y^2 - x^2 = c_2$

Let $u = y - x$, $v = y^2 - x^2$

$\frac{du}{dx} = -1$, $\frac{dv}{dx} = -2x$

Now, $p = \frac{dz}{dx}$ $\frac{du}{dy} = 1$ $\frac{dv}{dy} = 2y$

$= \frac{dz}{du}\frac{du}{dx} + \frac{dz}{dv}\frac{dv}{dx}$

$= \frac{dz}{du}\cdot(-1) + \frac{dz}{dv}(-2x)$

$= -\frac{dz}{du} - 2x\frac{dz}{dv}$ — (I)

$q = \frac{dz}{dy}$

$= \frac{dz}{du}\frac{du}{dy} + \frac{dz}{dv}\frac{dv}{dy}$

$= \frac{dz}{du} + 2y\frac{dz}{dv}$ — (II)

Now, $r = \frac{d}{dx}\left(\frac{dz}{dx}\right)$

$= \frac{d}{dx}\left(-\frac{dz}{du} - 2x\frac{dz}{dv}\right)$

$= -\frac{d^2z}{du^2}(-1) - \frac{d^2z}{dv\,du}(-2x)$

$- 2\frac{dz}{dv} - 2x\frac{d}{dx}\left(\frac{dz}{dv}\right)$

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$$= \frac{d^2z}{du^2} + 2x\frac{d^2z}{du\,dv} - 2\frac{dz}{dv} - 2x\left(\frac{d^2z}{dv^2}(-2x) + \frac{d^2z}{du\,dv}(-1)\right)$$

$$= \frac{d^2z}{du^2} + 2x\frac{d^2z}{du\,dv} - 2\frac{dz}{dv} + 4x^2\frac{d^2z}{dv^2} + 2x\frac{d^2z}{du\,dv} \quad \text{— (iii)}$$

$$t = \frac{d}{dy}\left(\frac{dz}{dy}\right)$$

$$= \frac{d}{dy}\left(\frac{dz}{du} + 2y\frac{dz}{dv}\right)$$

$$= \frac{d^2z}{du^2}\cdot 2y\frac{d^2z}{du\,dv} + 2\frac{dz}{dv} + 2y\left(\frac{d^2z}{dv^2}2y + \frac{d^2z}{du\,dv}\right) \quad \text{— (iv)}$$

$$s = \frac{d}{dy}\left(\frac{dz}{dx}\right)$$

$$= -\left\{\frac{d}{dy}\left(\frac{dz}{du} + 2x\frac{dz}{dv}\right)\right\}$$

$$= -\left\{\frac{d^2z}{du^2} + \frac{d^2z}{du\,dv}(2y) + 2x\left(\frac{d^2z}{dv^2}(2y) + \frac{d^2z}{du\,dv}\right)\right\}$$

$$= -\frac{d^2z}{du^2} - 2y\frac{d^2z}{du\,dv} - 4xy\frac{d^2z}{dv^2} - \frac{d^2z}{du\,dv} \quad \text{— (v)}$$

So, $yr + (x+y)s + xt = 0$

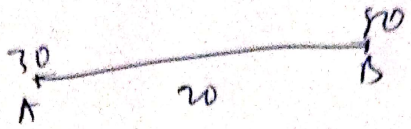
$$y\left(\frac{d^2z}{du^2} + 2x\frac{d^2z}{du\,dv} - 2\frac{dz}{du} + 4x^2\frac{d^2z}{dv^2} + 2x\frac{d^2z}{du\,dv}\right)$$

$$\Rightarrow y(r+s) + x(s+t) = 0$$

putting values from (iii), (iv), (v) gives the required soln

The ends A and B of a rod 20 cm long have the temperature at $30°C$ and $80°C$ until steady state prevails. The temperature of the ends are changed to $40°C$ and $60°C$ respectively. Find the temperature distribution in the rod are time t. (20)

Steady state, initial temp



$$u(x,0) = 30 + \frac{50}{20}(x)$$

$$u(0,t) = 40$$

$$u(2L,t) = 60$$

Find steady state temp

$$u_s(x) = 40 + x$$

Now, u can be divided as

$$u = u_s + u_t \qquad u_t = v_{transient}$$

u_t will also follow the heat equation.

$$u_t(0,t) = 0$$

$$u_t(L,t) = 0$$

$$u_t(x,0) = 30 + \frac{5}{2}x - 40 - x$$

$$= \frac{3}{2}x - 10$$

v_t satisfies the heat equation

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(d) $u_t = X(x)T(t)$

So, $\frac{du}{dt}, \frac{d^2u}{dx^2}$

$\Rightarrow XT' = c^2X''T$

$\Rightarrow \frac{T'}{T} = \frac{c^2X''}{X} = K$ same constant

K one is function of T & other is of X

Suppose $K = 0$

$\Rightarrow T' = 0$ & $X'' = 0$

$\Rightarrow T = k_1$ $\Rightarrow X = k_2x + k_3$

$x = 0, \Rightarrow 0$ (boundary)

$\Rightarrow k_3 = 0$

don't satisfy initial condition

Suppose $K = \lambda^2$

$\Rightarrow T' = \lambda^2 T$ $X'' = \frac{\lambda^2}{c^2}X$

$\Rightarrow T = Ae^{+\lambda^2 t}$ $\left(D^2 - \frac{\lambda^2}{c^2}\right)X = 0$

$A_1 e^{\frac{\lambda}{c}x} + B_1 e^{-\frac{\lambda}{c}x}$

But T keeps increasing indefinitely

So, not possible

Suppose $k = -\lambda^2$

Then $T = Ae^{-\lambda^2 t}$ $X = A\sin\frac{\lambda}{c}x + D\cos\frac{\lambda}{c}x$

$X(0) = 0$

$X(L) = 0$

$\Rightarrow D = 0$

So, $u = Ee^{-\lambda^2 t}\sin\frac{\lambda}{c}x$ where $E = AC$

Given $u_x(L) = 0$

$\Rightarrow \sin\frac{\lambda}{c}L = \sin n\pi$

$\Rightarrow \lambda = \frac{cn\pi}{L}$

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by principle of superposition

$$u(x,t) = \sum E_n e^{-\left(\frac{c^2 n^2 \pi^2}{L^2}\right)t} \sin\left(\frac{n\pi}{L}\right)x$$

At $t=0$

$$\sum E_n \sin\left(\frac{n\pi}{L}\right)x = \frac{3}{2}x - 10$$

So, $\frac{L}{2}E_n = \int_0^L \left(\frac{3}{2}x - 10\right)\sin\left(\frac{n\pi}{L}\right)x$

$$= \left(\frac{3}{2}x - 10\right)\left(-\cos\frac{n\pi}{L}x\right)\frac{L}{n\pi}\Big]_0^L + \int_0^L \frac{3}{2}\left(\cos\frac{n\pi}{L}x\right)\frac{L}{n\pi}$$

$$= \frac{-L}{n\pi}\left(\frac{3}{2}x - 10\right)\left(\sin\frac{n\pi}{L}x\right)\frac{L}{n\pi} + \frac{L^2}{n^2\pi^2}\frac{3}{2}\sin\frac{n\pi}{L}x \cdot \frac{L}{n\pi}\Big]_0^L$$

$$= -\left(\frac{L^2}{n\pi}\right)^n \times \frac{L}{2}E_n + \frac{3L^2}{2(n\pi)^2} \times$$

$$= -\frac{L}{n\pi}\left[\left(\frac{3}{2}L - 10\right)(-1)^n - (-10)\right] + \frac{3}{2}\frac{L}{2\pi}\sin\frac{n\pi}{L}x\,\frac{L}{n\pi}\Big]_0^L$$

$$\Rightarrow E_n = \frac{-2}{n\pi}\left[20(-1)^n + 10\right]$$

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And $u_2 u_3 + u_4$ is the required Solution

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