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Control Systems

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■■■■

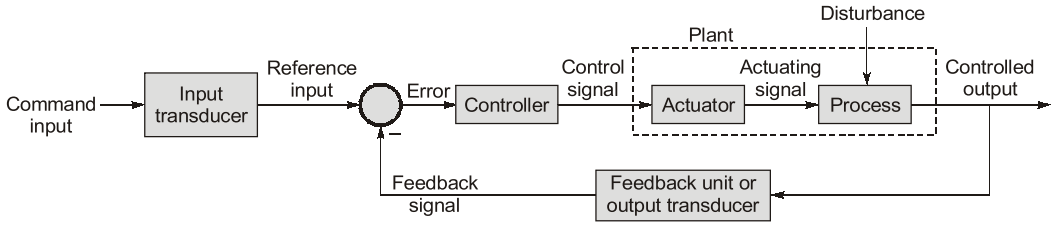
CHAPTER 1

Introduction

Control System:

- Control system is a means by which any quantity of interest in a machine, mechanism or other equipment is maintained or altered in accordance with a desired manner.
- Control system can also be defined as the combination of elements arranged in a planned manner wherein each element causes an effect to produce a desired output.

1.1 Elements of Control System (Components)



The following block-diagram representation shows typical elements of a control system.

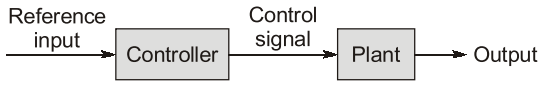
1. **Command input:** An input signal which, has a different form than that of feedback signal.
2. **Input transducer:** A transducer that converts the command signal into the same form as that of feedback signal.
3. **Reference input:** An external signal applied to the control system in order to command a specified action of plant.
4. **Error:** The algebraic sum of reference signal input and feedback signal which is fed to controller.
5. **Actuator:** A device that produces appropriate actuating signal that causes the necessary change in the controlled variable.
6. **Controller:** Produces control signal that is applied to actuator.
7. **Control signal:** Signal from the output of the controller that is applied to the plant to affect necessary changes in controlled variable.
8. **Process:** The central component whose output variable is to be controlled. The actuator and process are together known as plant.
9. **Feedback unit:** An output transducer that converts the controlled output into proper form of feedback signal.
10. **Feedback signal:** Generated by feedback unit and is a function of controlled output.

Control systems are classified into two general categories as open-loop and closed-loop systems.

NOTE: Open loop control systems does not require performance analysis.

1.2 Open-Loop Control System

A control system in which the output has no effect on the control action is called open loop control system. The output is neither measured nor fed back for comparison with the input and to produce an error signal.
Ex.: Washing machine, Coffee dispenser machine, Bread toaster.



For desired output, the system is calibrated (input is set at a predetermined value). For example, preset timings of traffic lights.

Advantages of Open-loop System

1. Simple in construction and easy to maintain
2. Less expensive due to minimum control devices
3. No instability problem
4. Accurate performance once calibrated
5. Convenient when output is difficult to measure or not economically feasible.

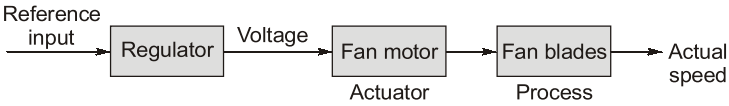
Disadvantages of Open-loop System

1. Disturbances cause drift in desired output
2. Changes in calibration causes error in system
3. Recuperation at regular intervals may be necessary to maintain quality of output.

***Example* - 1.1** Explain control action of a ceiling fan regulator and draw the component block diagram.

Solution:

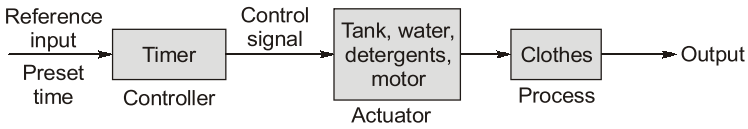
A fan regulator consists of a series switch and speed regulator which alters the voltage of fan motor for ataining different speeds. The output speed is not measured so the control scheme is open loop.



***Example* - 1.2** Indicate the type of control of an automatic washing machine and represent using a block diagram.

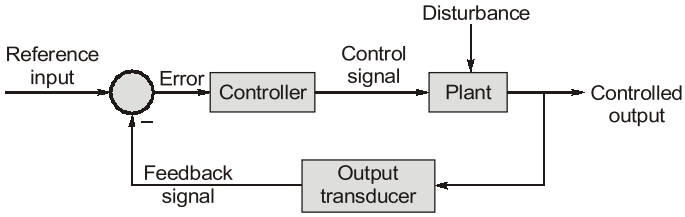
Solution:

An automatic washing machine has preset timings for various operations like soaking, washing, rinsing and drying. The timer and relay acts as controller. The clothes to be washed form the process. There is no measurement of quality of wash so it is an open-loop system.



1.3 Closed-Loop Control System

A control system that measures output and adjusts the input accordingly by using a feedback signal is called closed-loop control system. So, it also known as feedback control system.



Advantages of Closed Loop Control Systems

(a) Accurate and reliable
(b) Reduced effect of parameter variation
(c) Bandwidth of the system can be increased with negative feedback
(d) Reduced effect of non-linearities

Disadvantages of Closed Loop Control Systems

(a) The system is complex and costly
(b) System may become unstable
(c) Gain of the system reduces with negative feedback

Some common examples of closed loop control systems are:

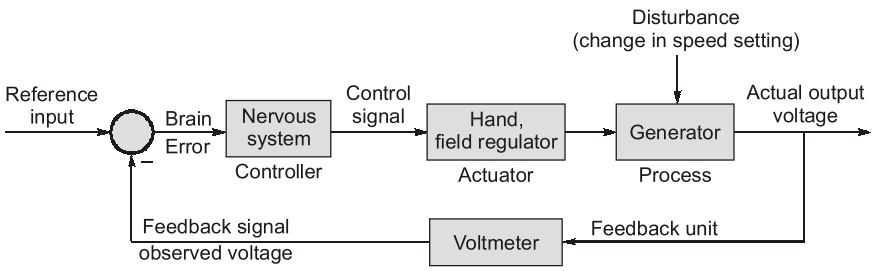
(a) Electric iron
(b) DC motor speed control
(c) A missile launching system (direction of missile changes with the location of moving target)
(d) Radar tracking system
(e) Human respiratory system
(f) Autopilot system
(g) Economic inflation

***Example* - 1.3** In a simple voltage control scheme for a dc generator running at constant speed, a human operator observes the voltmeter reading. If there is any error, the field current is adjusted to keep the terminal voltage at a specified value. Represent the above system by a block-diagram.

Solution:

For the given manual feedback control system:

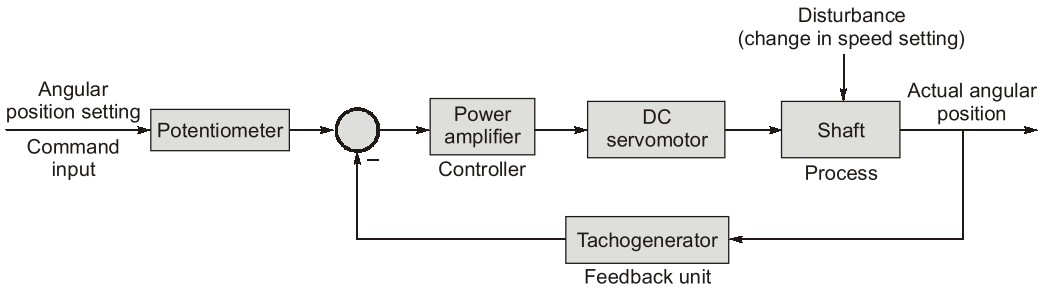
Reference input : Voltmeter leading
Error detector : Brain
Controller : Nervous system
Actuator : Hand, field regulator
Process : D.C. generator
Output transducer : Voltmeter



Example - 1.4 Discuss the position control scheme of a dc servomotor. Draw the block diagram and label the components.

Solution:

The position control scheme using a dc servomotor can be achieved by varying the armature voltage at constant field current. The angular position is preset on the potentiometer. The dc voltage given by tachogenerator is preoperational to angular rotation of shaft. This is compared with potentiometer output voltage to obtain error signal. The error is amplified and applied to dc motor and its angular position is varied.



- Feedback is not used for improving stability
- An open loop stable system may also become unstable when negative feedback is applied
- Except oscillators, in positive feedback, we have always unstable systems.

1.4 Comparison between Open-Loop and Closed-Loop Control Systems

Open-loop system	Closed-loop system
1. Internal or external disturbances cause drifts in desired output.	1. Use of feedback makes the system response relative insensitive to external and internal disturbances.
2. Accurate control is not possible unless regular recabrations are done.	2. Accurate control can be obtained using relatively inaccurate and inexpensive components.
3. Easier to build because system stability is not a major concert.	3. Stability is a major problem which may tend to overcorrect errors and cause oscillations.
4. Used for systems in which inputs are known ahead of time and there are no disturbances.	4. Advantageous only when unpredictable disturbance/parameter variations are present.
5. To decrease required cost and power of system, open-loop systems are used.	5. Due to more components used, closed-loop systems have higher and power.

1.5 Laplace Transformation

In order to transform a given function of time $f(t)$ into its corresponding Laplace transform first multiply $f(t)$ by e^{-st}, s being a complex number ($s = \sigma + j\omega$). Integrate this product with respect to time with limits from zero to ∞. This integration results in Laplace transform of f(t), which is denoted by $F(s)$ or $\mathcal{L}f[(t)]$.

The mathematical expression for Laplace transform is,

$$\mathcal{L}f[(t)] = F(s),\ t \geq 0$$

where,

$$F(s) = \int_0^\infty f(t).e^{-st}dt$$

The original time function f(t) is obtained back from the Laplace transform by a process called inverse Laplace transformation and denoted as $\mathcal{L}^{-1}$.

Thus, $\mathcal{L}^{-1}[Lf(t)] = \mathcal{L}^{-1}[F(s)] = f(t)$

The inverse Laplace transform can be calculated as follows:

$$L^{-1}(F(s)) = f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)\, e^{st}\, ds \qquad \text{for } t > 0$$

Existence of Laplace transform: The Laplace transform of a function $f(t)$ exists is the Laplace integral converges.

The time function $f(t)$ and its Laplace transform $F(s)$ form a transform pair.

S.No.	$f(t)$	$F(s) = L[f(t)]$
1.	$\delta(t)$ unit impulse at $t = 0$	1
2.	$u(t)$ unit step at $t = 0$	$\frac{1}{s}$
3.	$u(t-T)$ unit step at $t = T$	$\frac{1}{s}e^{-sT}$
4.	t	$\frac{1}{s^2}$
5.	$\frac{t^2}{2}$	$\frac{1}{s^3}$
6.	t^n	$\frac{n!}{s^{n+1}}$
7.	e^{at}	$\frac{1}{s-a}$
8.	e^{-at}	$\frac{1}{s+a}$
9.	te^{at}	$\frac{1}{(s-a)^2}$
10.	te^{-at}	$\frac{1}{(s+a)^2}$
11.	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
12.	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
13.	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
14.	$\sinh\omega t$	$\frac{\omega}{s^2-\omega^2}$
15.	$\cosh\omega t$	$\frac{s}{s^2-\omega^2}$
16.	$\frac{1}{b-a}(e^{-at}-be^{-bt})$	$\frac{1}{(s+a)(s+b)}$
17.	$\frac{1}{b-a}(be^{-bt}-ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
18.	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
19.	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Table of Laplace Transform Pairs

1.6 Properties and Theorems of Laplace Transforms

1. $L[Af(t)] = AF(s)$
2. $L[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3. $L\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0^+)$
4. $L\left[\frac{d^2}{dt^2} f(t)\right] = s^2F(s) - sf(0^-) - \frac{df}{dt}(0^-)$
5. $L\left[\frac{d^n}{dt^n} f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}(0^{\pm})$

   where, $f^{(k-1)}(t) = \frac{d^{k-1}}{dt^{k-1}} f(t)$
6. $L\left[\int f(t)\, dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)\, dt\right]_{t=0^-}$
7. $L\left[\int \ldots \int (t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n} \frac{1}{s^{n-k+1}} \left[\int \ldots . \int f(t)\,(dt)^n\right]_{t=0^-}$
8. $L\left[\int_0^t f(t)\, dt\right] = \frac{F(s)}{s}$
9. $\int_0^{\infty} f(t)\, dt = \lim_{s \to 0} F(s)$ if $\int_0^{\infty} f(t)\, dt$ exists
10. $L[e^{-at} f(t)] = F(s + a)$
11. $L[f(t-a)]\, u(t-a) = e^{-as} F(s)$ $\quad a \geq 0$ [Time shifting]
12. $L[t\, f(t)] = -\frac{dF(s)}{ds}$
13. $L[t^2 f(t)] = \frac{d^2}{ds^2} F(s)$
14. $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$ $\quad n = 1, 2, 3 \ldots .$
15. $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(s)\, ds$ if $\lim_{t \to 0} \frac{f(t)}{t}$ exists
16. $L\left[f\left(\frac{t}{a}\right)\right] = aF(as)$
17. $L\left[\int_0^t f_1(t-\tau)\, f_2(\tau)\, d\tau\right] = F_1(s)\, F_2(s)$
18. $L[f(t)\, g(t)] = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(p)\, G(s-p)\, dp$

19. $L[af_1(t) + bf_2(t)] = aF_1(s) + bF_2(s)$
20. Initial value theorem: $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$
21. Initial value theorem: $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

This theorem applies if and only if $\lim_{t\to 0} f(t)$ exists which is tree if all poles of $sF(s)$ are in the left half of the s-plane.

Example-1.5 Find the inverse Laplace transform of the following functions:

(i) $F(s) = \dfrac{s+2}{s^2+4s+6}$ (ii) $F(s) = \dfrac{5}{s(s^2+4s+5)}$

(iii) $F(s) = \dfrac{s^2+2s+3}{s^3+6s^2+12s+8}$ (iv) $F(s) = \dfrac{1}{(s+1)^2(s+2)}$

Solution:

(i) $$F(s) = \frac{s+2}{s^2+4s+6}$$

$\because$ The term $(s^2 + 4s + 6)$ can be expressed as $[(s+2)^2 + (\sqrt{2})^2]$

$\therefore$ $$F(s) = \frac{s+2}{(s+2)^2+(\sqrt{2})^2}$$

Taking inverse Laplace transform on both sides

$$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1} \frac{s+2}{(s+2)^2+(\sqrt{2})^2}$$

$\therefore$ $$f(t) = e^{-2t}\cos\sqrt{2}t\,u(t)$$

(ii) $$F(s) = \frac{5}{s(s^2+4s+5)}$$

Using partial fraction expansion,

$$\frac{5}{s(s^2+4s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+5}$$

The coefficients are determined as $A = 1$, $B = -1$ and $C = -4$

$\therefore$ $$F(s) = \frac{1}{s} - \frac{s+4}{s^2+4s+5}$$

$\because$ The term $(s^2 + 4s + 5)$ can be expressed as $[(s + 2)^2 + (1)^2]$

$\therefore$ $$F(s) = \frac{1}{s} - \frac{s+2}{[(s+2)^2+(1)^2]} - 2\frac{1}{[(s+2)^2+(1)^2]}$$

Taking inverse Laplace transform on both sides

$$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s+2}{(s+2)^2+(1)^2} - 2\frac{1}{(s+2)^2+(1)^2}\right]$$

or $$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1}\frac{1}{s} - \mathcal{L}^{-1}\frac{s+2}{[(s+2)^2+(1)^2]} - \mathcal{L}^{-1}\,2\frac{1}{[(s+2)^2+(1)^2]}$$

$\therefore$ $$f(t) = (1 - e^{-2t}\cos t - 2e^{-2t}\sin t)\,u(t)$$

(iii) $$F(s) = \frac{s^2+2s+3}{s^3+6s^2+12s+8}$$

$\because$ The denominator $(s^3 + 6s^2 + 12s + 8)$ can be expressed as $(s + 2)^3$

$\therefore$ $$F(s) = \frac{s^2+2s+3}{(s+2)^3}$$

Using partial fraction expansion

$$F(s) = \frac{A}{(s+2)} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}$$

The coefficients are determined as $A = 1$, $B = -2$ and $C = 3$

$\therefore$ $$F(s) = \frac{1}{s+2} - \frac{2}{(s+2)^2} + \frac{3}{(s+2)^3}$$

Taking inverse Laplace transform on both sides

$$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1}\left[\frac{1}{s+2} - \frac{2}{(s+2)^2} + \frac{3}{(s+2)^3}\right]$$

or $$\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1}\frac{1}{s+2} - \mathcal{L}^{-1}\frac{2}{(s+2)^2} + \mathcal{L}^{-1}\frac{3}{(s+2)^3}$$

$\therefore$ $$f(t) = \left[e^{-2t} - 2t\,e^{-2t} + \frac{3}{2}t^2\,e^{-2t}\right]\cdot u(t)$$

or $$f(t) = e^{-2t}\left[1 - t\left(2 - \frac{3}{2}t\right)\right]\cdot u(t)$$

(iv) The partial fraction expansion of $F(s)$ is

$$F(s) = A + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{s+2}$$

The coefficients A, B, C, D are given by

$$A = 0$$

$$B = \frac{d}{ds}(s+1)^2\,F(s)\Big|_{s=-1} = \frac{d}{ds}\left(\frac{1}{s+2}\right)\Big|_{s=-1} = -1$$

$$C = (s+1)^2\,F(s)\Big|_{s=-1} = \frac{1}{s+2}\Big|_{s=-1} = 1$$

$$D = (s+2)\,F(s)\Big|_{s=-2} = 1$$

Thus, $$F(s) = -\frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s+2}$$

$\Rightarrow$ $$f(t) = -e^{-t}\,u(t) + e^{-t}\,r(t) + e^{-2t}\,u(t)$$

Example - 1.6 Given, $F(s) = \dfrac{10}{s(s+1)}$, what is $\underset{t\to\infty}{Lim}\, f(t)$?

Solution:

$$F(s) = \frac{A}{s} + \frac{B}{(s+1)}$$

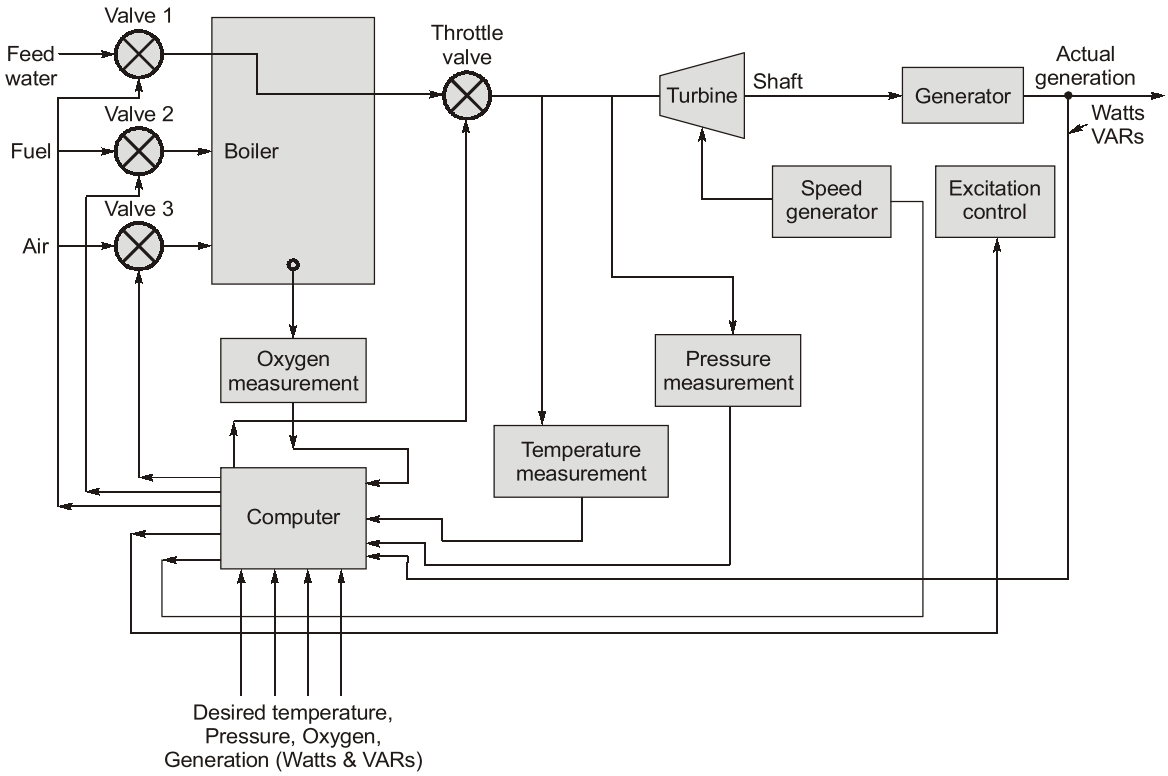
$$F(s) = \frac{10}{s} - \frac{10}{s+1}$$

Taking inverse Laplace,

$$f(t) = 10 - 10e^{-t}$$
$$= 10(1 - e^{-t})$$
$$\lim_{t \to \infty} f(t) = 10$$

Example - 1.7 Draw the boiler generator control of a typical thermal power station.

Solution:



Example - 1.8 Apply Laplace transform to solve ordinary differential equations, solve the following differential equation.

$$\frac{d^2x(t)}{dt^2} + \frac{3\,dx(t)}{dt} + 2x(t) = 15\,u(t); \quad x(0^+) = -1,\ x'(0^+) = 2$$

Solution:

Applying Laplace transform on both sides,

$$s^2X(s) - sx(0^+) - x'(0) + 3sX(s) - 3x(0^+) + 2X(s) = \frac{15}{s}$$

$$\Rightarrow \quad s^2X(s) + s - 2 + 3sX(s) + 3 + 2X(s) = \frac{15}{s}$$

$$\Rightarrow \quad X(s) = \frac{15 - s(s+1)}{s(s^2+3s+2)} = \frac{-s^2 - s + 15}{s(s+1)(s+2)}$$

Resolving into partial fractions we get,

$$X(s) = \frac{15}{2s} - \frac{15}{s+1} + \frac{13}{2(s+2)}$$

Taking inverse Laplace transform, we get

$$x(t) = \left[\frac{15}{2} - 15e^{-t} + \frac{13}{2}e^{-2t}\right] u(t)$$

Example - 1.9 A unit-doublet function is obtained by differentiating the unit-impulse function, expressed as,

$$u_2(t) = \lim_{t_0 \to 0} \frac{u(t) - 2[u(t - t_0)] + u(t - 2t_0)}{t_0^2}$$

Obtain Laplace transform of $u_2(t)$.

Solution:

$$L[u_2(t)] = \lim_{t_0 \to 0} \frac{1}{t_0^2}\left[\frac{1}{s} - \frac{2}{s}e^{-t_0 s} + \frac{1}{s}e^{-2t_0 s}\right]$$

$$= \lim_{t_0 \to 0} \frac{1}{t_0^2 s}\left[1 - 2\left(1 - t_0 s + \frac{t_0^2 s^2}{2} + \ldots\right) + \left(1 - 2t_0 s + \frac{4t_0^2 s^2}{2} + \ldots\right)\right]$$

$$= \lim_{t_0 \to 0} \frac{1}{t_0^2 s}[t_0^2 s^2 + \text{Higher order terms in } t_0 s] = s$$

Example - 1.10 Find the initial value of $\frac{df(t)}{dt}$ where $f(t)$ Laplace transform given by

$$F(s) = \frac{3s + 2}{s^2 + s + 1}$$

Solution:

Using initial value theorem,

$$f(0^+) = \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s) = \lim_{s \to \infty} \frac{s(3s+2)}{s^2+s+1} = 3$$

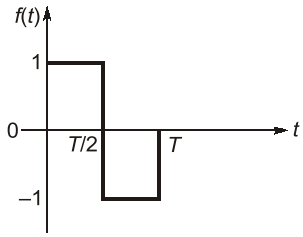
$$\therefore \quad L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^+)$$

$$= \frac{s(3s+2)}{s^2+s+1} - 3 = \frac{-s-3}{s^2+s+1}$$

The initial value of $\frac{d}{dt}(f(t))$ is given by

$$\lim_{t \to 0^+} \frac{df(t)}{dt} = \lim_{s \to \infty} s[sF(s) - f(0^+)] = \lim_{s \to \infty} \frac{-s^2 - 3s}{s^2 + s + 1} = -1$$

Example - 1.11 Find the Laplace transform of the following periodic function:



Solution:

$$\int_0^T f(t)e^{-st}\,dt = \int_0^{T/2} e^{-st}\,dt + \int_{T/2}^{T}(-1)\,e^{-st}\,dt$$

$$= \frac{e^{-st}}{-s}\Big|_0^{T/2} + (-1)\frac{e^{-st}}{-s}\Big|_{T/2}^{T} = \frac{e^{-sT/2}-1}{-s} + \frac{e^{-sT}-e^{-sT/2}}{s}$$

$$= \frac{1}{s}[e^{-sT} - 2e^{-sT/2} + 1] = \frac{1}{s}[1-e^{-sT/2}]^2$$

For periodic function, $$F(s) = \frac{\int_0^T f(t)e^{-st}\,dt}{1-e^{-sT}} = \frac{\frac{1}{s}(1-e^{-sT/2})^2}{1-e^{-sT}}$$

$$= \frac{1}{s}\frac{1-e^{-sT/2}}{1+e^{-sT/2}} = \frac{1}{s}\tanh\frac{sT}{4}$$

Derivation of Laplace transform for periodic functions:

$$L[f(t)] = \int_0^\infty f(t)e^{-st}\,dt = \sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} f(t)e^{-st}\,dt$$

By changing independent variable from t to τ, where $\tau = t - nT$,

$$L[f(t)] = \sum_{n=0}^{\infty} e^{-nTs}\int_0^T f(\tau)\,e^{-s\tau}\,d\tau \qquad \ldots(i)$$

Using, $f(\tau + nT) = f(\tau)$ since $f(t)$ is periodic with period T

$$\sum_{n=0}^{\infty} e^{-nTs} = 1 + e^{-Ts} + e^{-2Ts} + \ldots.$$

$$= 1 + e^{-Ts}(1 + e^{-Ts} + e^{-2Ts} + \ldots.)$$

$$= 1 + e^{-Ts}\sum_{n=\infty}^{\infty} e^{-nTs}$$

$$= 1 + e^{-Ts}\cdot\frac{1}{1-e^{-Ts}} = \frac{1}{1-e^{-Ts}}$$

$\therefore$ $$L[f(t)] = \frac{\int_0^T f(\tau)\,e^{-s\tau}\,d\tau}{1-e^{-Ts}}$$ [From equation (i)]

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